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The Low-Temperature Phase of Kac–Ising Models

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We analyze the low-temperature phase of ferromagnetic Kax–Ising models in dimensions $d \geqslant 2$. We show that if the range of interactions is γ^{-1}, then two disjoint translation-invariant Gibbs states exist if the inverse temperature β satisfies $\beta - 1 \geqslant \gamma^{\kappa}$, where $\kappa = d(1-\varepsilon)/(2d+2)(d+1)$, for any $\varepsilon > 0$. The proof involves the blocking procedure usual for Kac models and also a contour representation for the resulting long-range (almost) continuous-spin system which is suitable for the use of a variant of the Peierls argument.

KEY WORDS: Ising models; Kac potentials; low-temperature Gibbs states; contours; Peierls argument.

1. INTRODUCTION

In 1963 Kac *et al.*[1] introduced a statistical mechanical model of particles interacting via long, but finite-range interactions, i.e., through potentials of the form $J_\gamma(r) \equiv \gamma^d J(\gamma r)$, where J is some functional of bounded support or rapid decrease [the original example was $J(r) = e^{-r}$] and γ is a small parameter. These models were introduced as microscopic models for the van der Waals theory of the liquid–gas transition. In fact, in the context of these models it proved possible to derive in a mathematically rigorous way the van der Waals theory including the Maxwell construction in the limit $\gamma \downarrow 0$. In mathematical terms, this is stated as the Lebowitz–Penrose theorem[12]: The distribution of the density satisfies in the infinite-volume limit a large-deviation principle with a rate function that, in the limit as γ tends to zero, converges to the convex hull of the van der Waals free energy. For a review of these results, see, e.g., the textbook by Thompson.[15]

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Only rather recently has there been a more intense interest in the study of Kac models that went beyond the study of the global thermodynamic potentials in the Lebowitz–Penrose limit, but that also considers the distribution of *local mesoscopic* observables. This program has been carried out very nicely in the case of the Kac–Ising model in one spatial dimension by Cassandro *et al.*[6] A closely related analysis had been performed earlier by Bolthausen and Schmock.[4] These analyses can be seen as a rigorous derivation of a Ginzburg–Landau-type field theory for these models. Very recently, such an analysis was also carried out in a disordered version of the Kac–Ising model, the so-called Kac–Hopfield model.[1,2]

An extension of this work to higher dimensional situations would of course be greatly desirable. This turns out to be not trivial and, surprisingly, even very elementary question about the Kac model in $d\geqslant 2$ are unsolved. One of them is the natural conjecture that the critical inverse temperature $\beta_c(\gamma)$ in the Kac model should converge, as $\gamma\downarrow 0$, to the man-field critical temperature. This conjecture can be found, e.g., in a recent paper by Cassandro *et al.*[5] In that paper a lower bound $\beta_c(\gamma)\geqslant 1+b\gamma^2\,|\ln\gamma|$ is proven for $d=2$. A corresponding upper bound is only known in a very particular case where reflection positivity can be used.[3]

In addressing this question one soon finds the reason for this unfortunate state of affairs. All the powerful modern methods for analyzing the low-temperature phases of statistical mechanical models, such as low-temperature expansions and the Pirogov–Sinai theory, have been devised in view of models with short-range (often nearest neighbor) interactions, with possible longer range parts treated as some nuisance than be shown to be quite irrelevant. To deal with the genuinely long-range interaction in Kac models, that is, to exploit their long-range nature, these methods require substantial adaptation. The purpose of the present paper is to help to develop adequate techniques to deal with this problem—that beyond proving the conjecture of ref. 5 will, we hope, also provide a basis for the analysis of disordered Kac models. (Together with possible other means not touched by the presented paper; most notably with suitably developed expansion techniques for long-range models.)

The model we consider is defined as follows. We consider a measure space $(\mathscr{S},\mathscr{F})$ where $\mathscr{S}\equiv\{-1,1\}^{\mathbb{Z}^d}$ is equipped with the product topology of the discrete topology on $\{-1,1\}$ and $\mathscr{F}$ is the corresponding finitely generated sigma algebra. We denote an element of $\mathscr{S}$ by σ and call it a spin configuration. If $\Lambda\subset\mathbb{Z}^d$, we denote by σ_Λ the restriction of σ to Λ. For any finite volume Λ we define the energy of the configuration σ_Λ (given the external configuration σ_{Λ^c}) as

$$H_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\Lambda^c})\equiv-\tfrac{1}{2}\sum_{i,j\in\Lambda}J_\gamma(i-j)\,\sigma_i\sigma_j-\sum_{i\in\Lambda,\,j\notin\Lambda}J_\gamma(i-j)\,\sigma_i\sigma_j \tag{1.1}$$

where $J_\gamma(i) \equiv \gamma^d J(\gamma i)$ and $J: \mathbb{R}^d \to \mathbb{R}$ is a function that satisfies $\int_{\mathbb{R}^d} dx\, J(x) = 1$. For simplicity we will assume that J has bounded support, but the extension of our proof to more moderate assumptions on the decay properties of J is apparently not too difficult. To be completely specific we will even choose $J(r) \equiv c_d \mathbb{1}_{|x| \leqslant 1}$, where c_d normalizes the integral of J to one.[3] Here $|\cdot|$ is most conveniently chosen as the sup-norm on $\mathbb{R}^d$.

Finite-volume Gibbs measures ("local specifications") are defined as usual as

$$\mu^\eta_{\gamma,\beta,\Lambda}(\sigma_\Lambda) \equiv \frac{1}{Z^\eta_{\gamma,\beta,\Lambda}} \exp[-\beta H_{\gamma,\Lambda}(\sigma_\Lambda, \eta_{\Lambda^c})] \tag{1.2}$$

where $Z^\eta_{\gamma,\beta,\Lambda}$ is the usual partition function. Note that under our assumptions on J the local specifications for given Λ depend only on finitely many coordinates of η. An infinite-volume Gibbs state $\mu_{\gamma,\beta}$ is a probability measure on $(\mathcal{S}, \mathcal{F})$ that satisfies the DLR equations

$$\mu_{\gamma,\beta}\mu^{\cdot}_{\gamma,\beta,\Lambda} = \mu_{\gamma,\beta} \tag{1.3}$$

Our first result is the following.

Theorem 1. Let $d \geqslant 2$. Then there exists a function $f(\gamma)$ with $\lim_{\gamma\downarrow 0} f(\gamma) = 0$ such that for all $\beta > 1 + f(\gamma)$ there exist at least two disjoint extremal infinite-volume Gibbs states with local specifications given by (1.2). Moreover, for γ small enough, $f(\gamma) \leqslant \gamma^{(1-\varepsilon)/(2d+2)(1+1/d)}$ for arbitrary $\varepsilon > 0$.

Remark. This theorem shows that the conjecture of ref. 5 is correct. Since, as explained, e.g., in ref. 5, it follows from Dobrushin's uniqueness theorem[8] that $\beta_c(\gamma) \geqslant 1$ (in $d = 2$, ref. 5 proves that $\beta_c(\gamma) \geqslant 1 + b\gamma^2 |\ln \gamma|$], this implies that $\lim_{\gamma\downarrow 0} \beta_c(\gamma) = 1$ in the Kac model. While completing this work we received a paper Cassandro and Presutti[7] in which the conjecture of ref. 5 is also proven, but no explicit estimate on the asymptotics of the function $f(\gamma)$ is given. Their proof is rather different from ours. Although at the moment we make use of the spin-flip symmetry of the model, the contour language we introduce is also intended as a preparatory step for future use of the Pirogov–Sinai theory for nonsymmetric long-range models.

We will in fact get more precise information on the infinite-volume Gibbs measures in the course of the proof. This will be expressed in terms of the distribution of "local magnetization" $m_x(\sigma)$ defined on some suitable

[3] The generic name c_d will be used in the sequel for various finite, positive constants that only depend on dimension.

length scale $1 \ll l \ll \gamma^{-1}$. Given such a scale l, we will partition the lattice $\mathbb{Z}^d$ into blocks, denoted by x, of side length l. Identifying the block x with its label $x \in \mathbb{Z}^d$, we could thus set

$$x \equiv \{ i \in \mathbb{Z}^d \mid |i - lx| \leqslant l/2 \} \tag{1.4}$$

We then define for such blocks x

$$m_x(\sigma) \equiv \frac{1}{l^d} \sum_{i \in x} \sigma_i \tag{1.5}$$

In the sequel we will assume that all finite volumes we consider are compatible with these blocks, that is, are decomposable into them. We will also assume that γl is an integer. For any volume Λ compatible with the block structure we denote by $\mathcal{M}_\Lambda \subset \mathcal{F}_\Lambda$ the sigma algebra generated by the family of variables $\{m_x(\sigma)\}_{x \in \Lambda}$. The block variables will be instrumental in the proof of Theorem 1. However, they are also the natural variables to characterize the nature of typical configurations with respect to the Gibbs measure. We should note that this first step of passing to the variables $m_x(\sigma)$ is also used in ref. 7; in fact it is used in virtually all work on the Kac model.

The remainder of this article is organized as follows. In Section 2 the distribution of the block spins is formally introduced and the block-spin approximation of the Hamiltonian is discussed. In Section 3 we introduce our notion of Peierls contours and prove our theorem through a variant of the Peierls argument.[13]

2. BLOCK-SPIN APPROXIMATION

All the questions we want to answer in our model will after all concern the probabilities of events that are elements of the sigma algebras $\mathcal{M}_V$ for finite volumes V. If $\mathcal{A} \in \mathcal{M}_V$ is such an event and $\Lambda \supset V$, we have the following useful identity:

$$\begin{aligned} \mu^{\eta}_{\gamma,\beta,\Lambda}(\mathcal{A}) &= \sum_{\sigma_{\Lambda \setminus V}} \mu^{\eta}_{\gamma,\beta,\Lambda}(\sigma_{\Lambda \setminus V})\, \mu^{\sigma_{\Lambda \setminus V}, \eta_{\Lambda^c}}_{\gamma,\beta,V}(\mathcal{A}) \\ &= \sum_{\sigma_{\Lambda \setminus V}} \mu^{\eta}_{\gamma,\beta,\Lambda}(\sigma_{\Lambda \setminus V}) \sum_{\substack{m_x, x \in V \\ \{m_x\} \subset \mathcal{A}}} \mu^{\sigma_{\Lambda \setminus V}, \eta_{\Lambda^c}}_{\gamma,\beta,V}(\{m_x\}_{x \in V}) \end{aligned} \tag{2.1}$$

The sum over m_x runs of course over the values $\{-1, -1 + 2l^{-d}, ..., 1 - 2l^{-d}, 1\}$. Note that we may, if J has compact support, assume without loss of generality that Λ is sufficiently large so that the local specification

$\mu^{\sigma,1\backslash V,\eta,\Lambda^c}_{\gamma,\beta,V}$ does not depend on η. We will therefore drop the η in this expression.

The main point which makes the Kac model special is that the Hamiltonian is "close" to a function of the block spins. Namely, we may write

$$\begin{aligned}
H_{\gamma,V}(\sigma_V,\sigma_{V^c}) &= -\tfrac{1}{2}\sum_{x,y\in V}\sum_{i\in x,j\in y} J_\gamma(i,j)\,\sigma_i\sigma_j - \sum_{x\in V,y\in V^c}\sum_{i\in x,j\in y} J_\gamma(i,j)\,\sigma_i\sigma_j \\
&= -\tfrac{1}{2}\sum_{x,y\in V} J_\gamma(l(x-y))\sum_{i\in x,j\in y}\sigma_i\sigma_j \\
&\quad - \sum_{x\in V,y\in V^c} J_\gamma(l(x-y))\sum_{i\in x,j\in y}\sigma_i\sigma_j \\
&\quad -\tfrac{1}{2}\sum_{x,y\in V}\sum_{i\in x,j\in y}[J_\gamma(i-j)-J_\gamma(l(x-y))]\,\sigma_i\sigma_j \\
&= H^{(0)}_{\gamma,l,V}(m_V(\sigma_V),m_{V^c}(\sigma_{V^c})) + \Delta H_{\gamma,l,V}(\sigma_V,\sigma_{V^c})
\end{aligned}\tag{2.2}$$

where we have set [recall that $J_\gamma(lx)=l^{-d}J_{l\gamma}(x)$]

$$\begin{aligned}
H^{(0)}_{\gamma,l,V}(m_V,m_{V^c}) \equiv &-l^d\tfrac{1}{2}\sum_{x,y\in V} J_{\gamma l}(x-y)\,m_x m_y \\
&-l^d\sum_{x\in V,y\in V^c} J_{\gamma l}(x-y)\,m_x m_y
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
\Delta H_{\gamma,l,V}(\sigma_V,\sigma_{V^c}) = &-\tfrac{1}{2}\sum_{x,y\in V}\sum_{i\in x,j\in y}[J_\gamma(i-j)-J_\gamma(l(x-y))]\,\sigma_i\sigma_j \\
&-\sum_{x\in V,y\in V^c}\sum_{i\in x,j\in y}[J_\gamma(i-j)-J_\gamma(l(x-y))]\,\sigma_i\sigma_j
\end{aligned}\tag{2.4}$$

Lemma 2.1. For any $V\subset\mathbb{Z}^d$

$$\sup_\sigma |\Delta H_{\gamma,l,V}(\sigma_V,\sigma_{V^c})| \leqslant c_d\gamma l\,|V| \tag{2.5}$$

where c_d is some numerical constant that depends only on the dimension d.

Proof. This fact is well known and simple for all Kac models. In our case it follows from the observation that $[J_\gamma(i-j)-J_\gamma(l(x-y))]=0$ unless $|x-y|\approx 1/(\gamma l)$. ∎

As consequence of Lemma 2.1, we get the following useful upper and lower bounds for the distribution of the block spins:

$$\mu^{\sigma,\Lambda\setminus V}_{\gamma,\beta,V}(m_V) \lessgtr \frac{\exp[-\beta l^d H^{(0)}_{\gamma,l,V}(m_V, m_{V^c}) \prod_{x\in V} \mathbb{E}_\sigma \mathbb{1}_{m_x(\sigma)=m_x}}{\sum_{m_V} \exp[-\beta l^d H^{(0)}_{\gamma,l,V}(m_V, m_{V^c})] \prod_{x\in V} \mathbb{E}_\sigma \mathbb{1}_{m_x(\sigma)=m_x}} \times \exp(\pm\beta c_d \gamma l\,|V|) \tag{2.6}$$

Of course,

$$\mathbb{E}_\sigma \mathbb{1}_{m_x(\sigma)=m_x} \begin{cases} 2^{-l^d}\binom{l^d}{(1+m_x)\,l^d/2} & \text{if } \; l^d m_x/2\in\mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

and this, by Stirling's formula,

$$2^{-l^d}\binom{l^d}{(1+m_x)\,l^d/2} = e^{-l^d I(m_x) + O(\ln l)} \tag{2.8}$$

where $I(m)$, for $m\in[-1,1]$, is

$$I(m) = \frac{1+m}{2}\ln(1+m) + \frac{1-m}{2}\ln(1-m) \tag{2.9}$$

Therefore we define

$$\begin{aligned} E_{\gamma,\beta,l,V}(m_V, m_{V^c}) \equiv & -\tfrac{1}{2}\sum_{x,y\in V} J_{\gamma l}(x-y)\,m_x m_y \\ & -\sum_{x\in V,\,y\in V^c} J_{\gamma l}(x-y)\,m_x m_y + \beta^{-1}\sum_{x\in V} I(m_x) \end{aligned} \tag{2.10}$$

to get the following result.

Lemma 2.2. For any finite volume V and any configuration m_V we have

$$\mu^{\sigma,\Lambda\setminus V}_{\gamma,\beta,V}(m_V) \lessgtr \frac{\exp[-\beta l^d E_{\gamma,\beta,l,V}(m_V, m_{V^c}(\sigma_{V^c}))]}{\sum_{m_V} \exp[-\beta l^d E_{\gamma,\beta,l,V}(m_V, m_{V^c}(\sigma_{V^c}))]} \exp(\pm\beta c_d \gamma l\,|V|) \tag{2.11}$$

Remark. l will be chosen as tending to infinity as γ tends to zero. The idea is that that $E_{\gamma,\beta,l,V}$ is in a sense a "rate function"; that is, $E_{\gamma,\beta,l,V}$ alone determines the measures since the residual entropy is only of the order $[(d \ln l)/l^d]\,|V|$. The problem is that this is only meaningful when we consider events $\mathscr{A}$ for which the minimal $E_{\gamma,\beta,l,V}$ is of order $|V|$ above the ground state to make sure that neither the residual entropy nor the error terms in (2.11) invalidate the result. We will have to work in the next section to define such events.

It is instructive to rewrite the functional $E_{\gamma,\beta,l,V}$ in a slightly different form using that $-m_x m_y = \frac{1}{2}(m_x - m_y)^2 - \frac{1}{2}(m_x^2 + m_y^2)$ (we drop the indices γ, β, l henceforth, but keep this dependence in mind). We set

$$\tilde{E}_V(m_V, m_{V^c}) \equiv \frac{1}{4} \sum_{x,\,y \in V} J_{\gamma l}(x-y)(m_x - m_y)^2 + \frac{1}{2} \sum_{x \in V,\, y \in V^c} J_{\gamma l}(x-y)(m_x - m_y)^2 + \sum_{x \in V} f_\beta(m_x) \tag{2.12}$$

where f_β is the well-known free energy function of the Curie–Weiss model,

$$f_\beta \equiv [\beta^{-1} I(m_x) - \tfrac{1}{2} m_x^2] \tag{2.13}$$

Then

$$E_V(m_V, m_{V^c}) = \tilde{E}_V(m_V, m_{V^c}) - C_V(m_{V^c}) \tag{2.14}$$

where

$$C_V(m_{V^c}) \equiv \frac{1}{2} \sum_{x \in V,\, y \in V^c} J_{\gamma l}(x-y)\, m_y^2 \tag{2.15}$$

depends only on the variables on V^c.

The form $\tilde{E}_V$ makes it nicely evident that the energy functional favors configurations that are constant and close to the minima of the Curie–Weiss function $f_\beta(m)$.

3. PEIERLS CONTOURS

In this section we define an appropriate notion of Peierls contours in our model and use this to prove Theorem 1 by a version of the Peierls argument.[4] The general spirit behind the definition of Peierls contours can be loosely characterized as follows: We want to define a family of local events that have the property that at least one of them has to occur if the effect of boundary conditions does not propagate to the interior of the

[4] While the proof of ref. 7 is also based on a Peierls argument, their definition of Peierls contours is completely different from ours.

system. Then one must show that the probability that any of these events occurs is small. This requires in particular that the excess energy associated with such an event must be bounded from below uniformly with respect to what happens away from where the event is localized (in contrast, e.g., to the event $\sigma_0 = -1$, whose local energy depends on the configuration surrounding the point 0). It is customary to call such events *Peierls contours.* We will define contours in terms of the *block spin variables* $m_x(\sigma)$. More precisely, since it is crucial for us to exploit that the new interaction is still long range,[5] contours will be defined in terms of the *local averages* $\phi_x(m)$ and the *local variances* $\psi_x(m)$ defined through

$$\phi_x(m) \equiv \sum_y J_{\gamma l}(x-y)\, m_y \tag{3.1}$$

$$\psi_x(m) \equiv \sum_y J_{\gamma l}(x-y)(m_y - \phi_y(m))^2 \tag{3.2}$$

Then define the set

$$\tilde{\Gamma} \equiv \{x \mid \, ||\phi_x(m)| - m^*(\beta)| > \zeta m^*(\beta) \text{ or } \psi_x(m) > (\zeta m^*(\beta))^2\} \tag{3.3}$$

where $m^*(\beta)$ is the largest solution of the equation $x = \tanh \beta x$, that is, the location of the nonnegative minimum of the function f_β. We recall (see, e.g., ref. 10) that $m^*(\beta) = 0$ if $\beta \leqslant 1$, $m^*(\beta) > 0$ if $\beta > 1$, $\lim_{\beta \uparrow \infty} m^*(\beta) = 1$, and $\lim_{\beta \downarrow 1} [(m^*(\beta))^2/3(\beta - 1)] = 1$. To simplify notation we will write $m^* \equiv m^*(\beta)$ in the sequel. $\zeta < 1$ will be chosen in a suitable way later. Note that if the boundary conditions are such that, say, $\phi_x(m(\eta)) \approx -m^*$, then, if the configuration near the origin is such that $\phi_0(m(\sigma)) < 0$, there must be a region enclosing the origin on which ϕ takes the value zero and thus belongs to $\tilde{\Gamma}$. For a reason that will become clear later, in a first step we will regularize this set. For this we introduce a second blocking of the lattice, this time on the *scale of the range of the interaction.* The points u of this lattice are identified with the blocks

$$u \equiv \{x \in \mathbb{Z}^d \mid |x - u|(\gamma l)| \leqslant 1/(2\gamma l)\} \tag{3.4}$$

just as in (1.4). We write in a natural way $u(x)$ for the label of the unique block that contains x. We will call sets that are unions of such blocks u *regular sets.* We put

$$\Gamma_0 \equiv \{x \mid u(x) \cap \tilde{\Gamma} \neq \varnothing\} \tag{3.5}$$

[5] For that reason it is not possible to use directly the methods developed in ref. 9 for studying low-temperature phases of short-range continuous-spin models, although some of the ideas in that paper are used in our proof.

For some positive integer $n \geqslant 1$ to be chosen later, we now set

$$\underline{\Gamma} \equiv \{x \mid \operatorname{dist}(x, \underline{\Gamma}_0) \leqslant n(\gamma l)^{-1}\} \tag{3.6}$$

where dist is the metric induced by the sup-norm on $\mathbb{R}^d$. The integer n will depend on β and diverge as $\beta \downarrow 1$. The precise value of n will be specified later in (3.48). Notice that this definition assures that the set $\underline{\Gamma}$ is a regular set in the sense defined above. Connected components of the set $\underline{\Gamma}$ together with the specification of the values of m_x, $x \in \underline{\Gamma}$, are called contours and are denoted by Γ. For such a contour we introduce the notion of its *boundary* Γ in the following sense:

$$\partial\underline{\Gamma} \equiv \{x \in \underline{\Gamma} \mid \operatorname{dist}(x, \underline{\Gamma}^c) \leqslant n(\gamma l)^{-1}\} \tag{3.7}$$

Note that by our definition of Γ we are assured that $\partial\underline{\Gamma} \cap \underline{\Gamma}_0 = \varnothing$. We denote

$$D^{\pm} \equiv \{x \mid |\phi_x(m) \mp m^*| \leqslant \zeta m^*\} \cap \underline{\Gamma}^c \tag{3.8}$$

and call these regions $\pm$-correct. Each connected component of the boundary of Γ connects either to D^+ or D^-. We will denote such connected components by $\partial\underline{\Gamma}_i^+$ and $\partial\underline{\Gamma}_i^-$, respectively.

For a connected set $\underline{\Gamma}$ we denote by $\operatorname{int}\underline{\Gamma}$ the simply connected set obtained by "filling up the holes" of $\underline{\Gamma}$. This set is called the interior of a contour. The boundary of $\operatorname{int}\underline{\Gamma}$ will be referred to as the exterior boundary of $\underline{\Gamma}$. The connected component of $\partial\underline{\Gamma}$ that is also the boundary of $\operatorname{int}\underline{\Gamma}$ will be called the exterior boundary of Γ and denoted by $\partial\underline{\Gamma}^{\mathrm{ext}}$.

The strategy to prove Theorem 1 is the usual one. First we observe that if boundary conditions are strongly plus, then in order to have that, say, $|\phi_0(m) - m^*| > \zeta m^*$, it must be true that there exists a contour Γ such that $0 \in \operatorname{int}\underline{\Gamma}$. Thus it suffices to prove that the probability of contours is sufficiently small. This will require a *lower bound* on the energy of *any configuration compatible with the existence of* $\underline{\Gamma}$, and an *upper bound* on a carefully chosen *reference configuration* in which the contour is absent. We will show later (Lemma 3.8) that a lower bound on the energy can easily be given in terms of the functions ϕ and ψ, a fact that motivates the definition of $\tilde{\underline{\Gamma}}$. The long-range nature of the interaction and the fact that the m_x are essentially continuous variables require the construction of the extensive "safety belts" around this set in order to assure an effective decoupling of the core of a contour from its exterior. The crucial reason for the definition of contours through the nonlocal functions ϕ and ψ is, however, the fact that these are "slowly varying" functions of x for any configuration m.

Therefore, even if the core $\underline{\tilde{\Gamma}}$ is very "thin" (e.g., a single point), one can show that on a much larger set $|\phi_0(m) - m^*|$ or $\psi_x(m)$ must still be quite large (e.g., half of what is asked for in $\underline{\tilde{\Gamma}}$). This guarantees that in spite of the very thick "safety belts" we must construct around $\underline{\tilde{\Gamma}}$, the energy of a contour compares nicely with its volume $|\underline{\Gamma}|$.

We will now establish the "decoupling" properties. For this we must establish some properties of the configuration m on $\partial\underline{\Gamma}$ that minimizes $E_{\partial\underline{\Gamma}}$ for given boundary conditions.

Definition 3.1. A configuration m_V^{opt} is called optimal if m^{opt} minimizes $E_V(m_V, m_{V^c})$ for a given configuration m_{V^c}.

An important point is that away from $\underline{\tilde{\Gamma}}$, due to our definition of contours, configurations must be close to constant in the following sense:

Lemma 3.2. Assume that $x \notin \underline{\tilde{\Gamma}}$. Then:

(i) If $\operatorname{dist}(x, \underline{\tilde{\Gamma}}) > 1/(\gamma l)$,

$$\sum_y J_{\gamma l}(x-y)(m_y \pm m^*)^2 \leqslant 4\zeta^2(m^*)^2 \tag{3.9}$$

(ii) For any $V \subset \underline{\tilde{\Gamma}}^c$

$$\sum_{y \in V} J_{\gamma l}(x-y)\, |m_y \pm m^*| \leqslant 2\zeta m^* \left[\sum_{y \in V} J_{\gamma l}(x-y) \right]^{1/2} \tag{3.10}$$

where the sign depends on whether $\phi_x(m)$ is positive or negative in the region.

Proof. The proof of (3.9) is straightforward from the definition of $\underline{\tilde{\Gamma}}$ in (3.3), and (3.10) follows from (3.9) by the Schwartz inequality. ∎

We will now establish properties of an optimal configuration on regular sets with boundary conditions that satisfy properties (3.9) and (3.10).

Lemma 3.3. Let V be a regular set. Then there exists $\zeta_d > 0$ depending only on the dimension d such that if m_{V^c} is a boundary condition of $(+)$ type for which (3.9) and (3.10) hold with $\zeta \leqslant \zeta_d$, then for all $x \in V$, $|m_x^{\text{opt}} - m^*| \leqslant m^*/2$. The corresponding statement holds for $(-)$-type boundary conditions.

Proof. We see from (2.10) that we must have, for $y \in V$,[6]

$$0 = \frac{d}{dm_y} E_V(m_V, m_{V^c}) = \beta^{-1} I'(m_y) - \phi_y(m) \tag{3.11}$$

(3.11) can be written as

$$m_y = \tanh(\beta \phi_y(m)) \tag{3.12}$$

We may tacitly assume that $\phi_y(m)$ is positive (this assumption will be shown to be consistent). Since m^* in a stable fixed point of the function $\tanh \beta m$ that attracts all points on the positive half-line, it follows that $|\tanh(\beta\phi_y(m)) - m^*| \leqslant |\phi_y(m) - m^*|$ and in particular, if $\phi_y(m) < m^*$, $\tanh(\beta\phi_y(m)) > \phi_y(m)$, while for $\phi_y(m) > m^*$, $\tanh(\beta\phi_y(m)) < \phi_y(m)$. We will first show that $m_x^{\mathrm{opt}} \geqslant m^*/2$. Let $x \in V$ denote a point where

$$m_x = \inf_{y \in V} \{ m_y \mid m_y \leqslant m^* \} \tag{3.13}$$

If $m_x = m^*$, there is nothing to prove. But if $m_x < m^*$, then (3.13) can only be satisfied if $\operatorname{dist}(x, \partial V) < 1/(\gamma l)$. For such points we can write

$$\begin{aligned} m_x - m^* &\geqslant \sum_{y \in V} J_{\gamma l}(x-y)(m_y - m^*) + \sum_{y \in V^c} J_{\gamma l}(x-y)(m_y - m^*) \\ &\geqslant (m_x - m^*) \sum_{y \in V} J_{\gamma l}(x-y) - 2\zeta m^* \left[\sum_{y \in V^c} J_{\gamma l}(x-y) \right]^{1/2} \end{aligned} \tag{3.14}$$

where the second line follows by (3.10). Hence

$$m_x - m^* \geqslant -\frac{2\zeta m^*}{\left[\sum_{y \in V^c} J_{\gamma l}(x-y)\right]^{1/2}} \tag{3.15}$$

On the other hand, (3.14) holds for any other point $y \in V$ as well, and inserting this into the first line of (3.14), we get

$$m_x - m^* \geqslant (m_x - m^*) \sum_{y \in V} \sum_{z \in V} J_{\gamma l}(x-y) J_{\gamma l}(y-z) - 4\zeta m^* \tag{3.16}$$

[6] We ignore the fact that m_x takes only discrete values and look for the optimal solution in the space of real-valued m. The point is that given such a solution, a discrete-valued approximation can be constructed that differs in energy by less than $|\Gamma|/l^d$, which is negligibly small.

Clearly, we have won if either

$$1-\sum_{y\in V}\sum_{z\in V} J_{\gamma l}(x-y)\,J_{\gamma l}(y-z)\geqslant 8\zeta \tag{3.17}$$

or

$$\left[\sum_{y\in V^c} J_{\gamma l}(x-y)\right]^{1/2}\geqslant 4\zeta \tag{3.18}$$

Due to the fact that V is composed of cubes of side length of the range of the interaction, this follows from simple considerations if ζ is smaller than some dimension-dependent constant. (Here is the reason for our definition of $\underline{\Gamma}_0$.) In fact,

$$\begin{aligned}1&-\sum_{y\in V}\sum_{z\in V} J_{\gamma l}(x-y)\,J_{\gamma l}(y-z)\\&=\sum_{y\in V^c} J_{\gamma l}(x-y)+\sum_{y\in V}\sum_{z\in V^c} J_{\gamma l}(x-y)\,J_{\gamma l}(y-z)\end{aligned}\tag{3.19}$$

The point is that the second term on the right-hand side (3.19) cannot be too small as long as $\operatorname{dist}(x, V^c)\leqslant 1/(\gamma l)$, for regular V (if V is *not* regular, this statement does not hold, of course; just consider a thin, long spike entering into V and let x be near the tip of the spike!). In fact, that worst situation here occurs if x is at a distance $r/(\gamma l)$ from a "corner" of V^c. One easily verifies that even in this case

$$\begin{aligned}&\sum_{y\in V}\sum_{z\in V^c} J_{\gamma l}(x-y)\,J_{\gamma l}(y-z)\\&\qquad\geqslant 2^{-(d+1)}\int_0^1 ds\,(r+s)^{d-1}(1-s)^d\\&\qquad\geqslant 2^{-(d+1)}\int_0^1 ds\,s^{d-1}(1-s)^d\\&\qquad=2^{-(d+2)}\frac{((d-1)!)^2}{(2d-1)!}\end{aligned}\tag{3.20}$$

so that (3.18) is verified if 4ζ is smaller than this number. The numerical value of that bound can of course be improved, but we do not seek to do that.

Having established that $m_x \geqslant m^*/2$ in V, a trivial computation shows that our starting assumption that $\phi_x(m) > 0$ is also verified. Thus we have proven that $m_x^{\mathrm{opt}} \geqslant m^*/2$. In the same way one shows also that $m_x^{\mathrm{opt}} \leqslant 3m^*/2$, which concludes the proof of the lemma. ∎

In the sequal the notion of n-layer set defined in the following definition will be convenient.

Definition 3.4. A regular set V is called an n-layer annulus if it is of the form

$$V = \{x \in \tilde{V}^c \mid \operatorname{dist}(x, \tilde{V}) \leqslant n(\gamma l)^{-1}\} \tag{3.21}$$

for some connected set $\tilde{V}$ that is composed of blocks u. The sets

$$V_k \equiv \{x \in \tilde{V}^c \mid (k-1)(\gamma l)^{-1} < \operatorname{dist}(x, \tilde{V}) \leqslant k(\gamma l)^{-1}\} \tag{3.22}$$

are called the kth layers of V.

Note that the sets $\partial\underline{\Gamma}$ are by their definition n-layer sets.

We are interested in some properties of optimal configurations on n-layer sets. For this we will use the following simple fact about the function f_β, which may be found, e.g., in ref. 2:

Lemma 3.5. Let $f_\beta(m) = \beta^{-1} I(m) - \frac{1}{2}m^2$. Then, for all $m \in [-1, 1]$,

$$f_\beta(m) - f_\beta(m^*) \geqslant c(\beta)(|m| - m^*)^2 \tag{3.23}$$

where

$$c(\beta) \equiv \frac{\ln\cosh(\beta m^*)}{\beta(m^*)^2} - \frac{1}{2} \tag{3.24}$$

has the property that $c(\beta) > 0$ for all $\beta > 1$ and

$$\lim_{\beta \downarrow 1} \frac{c(\beta)}{(m^*)^2} = \frac{1}{12} \tag{3.25}$$

From this we will derive the following lemma (the analog of this lemma for short-range and purely quadratic Hamiltonians appeared already in ref. 9).

Lemma 3.6. Let V be an n-layer set with $n \geqslant r/c(\beta)$. Then there exists a layer V_k in V such that

$$\sum_{x \in V_k} (m_x^{\text{opt}} \mp m^*)^2 \leqslant 2^{-r} \tfrac{1}{8} (m^*)^2 (|V_1| + |V_n|) \tag{3.26}$$

with $\mp$ depending on the type of boundary conditions.

Proof. We assume that the boundary conditions are of $(+)$ type. Note that by Lemma 3.3, we know that for $x \in V$, $m_x^{\text{opt}} = |m_x^{\text{opt}}|$. Let us set $u_x \equiv |m_x| - m^*$ and and use the abbreviation

$$\|u_{V_k}\|_2^2 \equiv \sum_{x \in V_k} (u_x)^2 \tag{3.27}$$

and analogously for other functions. Then it is obvious from (2.12) that for any configuration

$$\tilde{E}_{V \backslash V_1 \backslash V_2}(m_{V \backslash V_1 \backslash V_n}, m_{V_1 \cup V_n}) \geqslant \sum_{k=2}^{n-1} c(\beta) \|u_{V_k}\|_2^2 + \sum_{x \in V \backslash V_1 \backslash V_n} f_\beta(m^*) \tag{3.28}$$

On the other hand, we may consider a configuration that equals m^{opt} on V_1 and V_n and has $m_x = m^*$ for all $x \in V \backslash V_1 \backslash V_n$. For this configuration

$$\begin{aligned} &\tilde{E}_{V \backslash V_1 \backslash V_2}(m_{V \backslash V_1 \backslash V_n} = m^*, m^{\text{opt}}_{V_1 \cup V_n}) \\ &\quad = \tfrac{1}{2} \sum_{\substack{x \in V \backslash V_1 \backslash V_n \\ y \in V_1 \cup V_n}} J_{\gamma l}(x-y)(m_y^{\text{opt}} - m^*)^2 + \sum_{x \in V \backslash V_1 \backslash V_n} f_\beta(m^*) \end{aligned} \tag{3.29}$$

By the definition of m^{opt}, it must thus be true that

$$\begin{aligned} 0 &\geqslant \tilde{E}_{V \backslash V_1 \backslash V_2}(m^{\text{opt}}_{V \backslash V_1 \backslash V_n}, m^{\text{opt}}_{V_1 \cup V_n}) \\ &\quad - \tilde{E}_{V \backslash V_1 \backslash V_2}(m^{\text{opt}}_{V \backslash V_1 \backslash V_n} = m^*, m^{\text{opt}}_{V_1 \cup V_n}) \\ &\geqslant \sum_{k=2}^{n-1} c(\beta) \|u_{V_k}\|_2^2 - \tfrac{1}{2} \sum_{\substack{x \in V \backslash V_1 \backslash V_n \\ y \in V_1 \cup V_n}} J_{\gamma l}(x-y)(m_y^{\text{opt}} - m^*)^2 \\ &\geqslant \sum_{k=2}^{n-1} c(\beta) \|u_{V_k}\|_2^2 - \tfrac{1}{2}(\|u^{\text{opt}}_{V_{-1}}\|_2^2 + \|u^{\text{opt}}_{V_n}\|_2^2) \end{aligned} \tag{3.30}$$

Thus, for any $q<n/2$, we have

$$
\begin{aligned}
qc(\beta) \inf_{k=2}^{q+1} & [\|u^{\text{opt}}_{V_k}\|_2^2 + \|u^{\text{opt}}_{V_{n+1-k}}\|_2^2] \\
&\leqslant \sum_{k=2}^{n-1} c(\beta)[\|u^{\text{opt}}_{V_k}\|_2^2 + \|u^{\text{opt}}_{V_{n+1-k}}\|_2^2] \\
&\leqslant \tfrac{1}{2}\|u^{\text{opt}}_{V_1}\|_2^2 + \tfrac{1}{2}\|u^{\text{opt}}_{V_1}\|_2^2
\end{aligned}
\tag{3.31}
$$

from where

$$
\inf_{k=2}^{q+1} [\|u_{V_k}\|_2^2 + \|u_{V_{n+1-k}}\|_2^2] \leqslant \frac{1}{2qc(\beta)} [\|u^{\text{opt}}_{V_1}\|_2^2 + \|u^{\text{opt}}_{V_1}\|_2^2] \tag{3.32}
$$

If q is chosen as the smallest integer greater than $1/c(\beta)$, this shows that there exists $2 \leqslant k \leqslant q+1$ such that

$$
[\|u_{V_k}\|_2^2 + \|u_{V_{n+1-k}}\|_2^2] \leqslant \tfrac{1}{2}[\|u^{\text{opt}}_{V_1}\|_2^2 + \|u^{\text{opt}}_{V_1}\|_2^2] \tag{3.33}
$$

Iterating this construction and using that by Lemma 3.3

$$
\tfrac{1}{2}\|u^{\text{opt}}_{V_1}\|_2^2 + \tfrac{1}{2}\|u^{\text{opt}}_{V_1}\|_2^2 \leqslant \tfrac{1}{8}(m^*)(|V_1| + |V_n|) \tag{3.34}
$$

we arrive at the assertion of the lemma. ∎

We are now ready to construct our reference configuration and give an upper bound on its energy. For given contour Γ and compatible external configuration m on $\underline{\Gamma}^c$ and on the core $\tilde{\underline{\Gamma}}$ we call m^{opt} the configuration on $\underline{\Gamma}$ that minimizes the energy under these boundary conditions. Clearly such a configuration is also an optimal configuration on $\partial\underline{\Gamma}$ in the sense of Definition 3.1. Thus by Lemma 3.2 we know that in each connected component $\partial\underline{\Gamma}_i^\pm$ of the boundary of Γ there exists a layer $\mathscr{L}_i^\pm$ of thickness $1/(\gamma l)$ in $\partial\Gamma_i^\pm$ such that

$$
\|m^{\text{opt}}_{\mathscr{L}} \mp m^*\|_2^2 \leqslant 2^{-r}\tfrac{1}{8}(m^*)^2 [|V_1(\partial\underline{\Gamma}_i^\pm)| + |V_n(\partial\underline{\Gamma}_i^\pm)|]
$$

For given $\mathscr{L}_i^\pm$ we decompose $\partial\underline{\Gamma}_i^\pm$ into the two sets

$$
\partial\underline{\Gamma}_{i,in}^\pm \equiv \{x \in \partial\underline{\Gamma}_i^\pm \backslash \mathscr{L}_i^\pm \mid \operatorname{dist}(x, D^\pm) > \operatorname{dist}(\mathscr{L}_i^\pm, D^\pm)\} \tag{3.35}
$$

$$
\partial\underline{\Gamma}_{i,\text{out}}^\pm \equiv \partial\underline{\Gamma}_i^\pm \backslash \partial\underline{\Gamma}_{i,in}^\pm \tag{3.36}
$$

Without loss of generality we assume that the exterior boundary of our contour is attached to the +-correct region. We now define the reference configuration m^{ref},

$$m_x^{\text{ref}} \equiv \begin{cases} m_x^{\text{opt}} & \text{if} \quad x \in \partial\underline{\Gamma}_{i,\text{out}}^{+} \\ -m_x^{\text{opt}} & \text{if} \quad x \in \partial\underline{\Gamma}_{i,\text{out}}^{-} \\ m^* & \text{for all other } x \in \underline{\Gamma} \\ m_x & \text{for } x \in D^+ \\ -m_x & \text{for } D^- \end{cases} \tag{3.37}$$

Lemma 3.7. Let m^{ref} be defined in (3.37). Then for any compatible external configuration we have that

$$\begin{aligned} \tilde{E}_{\underline{\Gamma}}(m_{\underline{\Gamma}}^{\text{ref}}, m_{\underline{\Gamma}^c}^{\text{ref}}) \geqslant & \sum_{i, \pm} \tilde{E}_{\partial\underline{\Gamma}_{i,\text{out}}^{\pm}}(m_{\partial\underline{\Gamma}_{i,\text{out}}^{\pm}}^{\text{opt}}, m_{\underline{\Gamma}^c}^{\text{ref}}) \\ & + \sum_{i, \pm} 2^{-r} \tfrac{1}{8}(m^*)^2 [\,|V_1)\partial\underline{\Gamma}_i^{\pm})| + |V_n(\partial\underline{\Gamma}_i^{\pm})|\,] \\ & + \sum_{x \in \underline{\Gamma} \setminus \partial\underline{\Gamma}_{\text{out}}} f_\beta(m^*) \end{aligned} \tag{3.38}$$

Proof. The proof of this estimate is obvious from the definition of m^{ref} and Lemma 3.6. Note that in the terms

$$\tilde{E}_{\partial\underline{\Gamma}_{i,\text{out}}^{\pm}}(m_{\partial\underline{\Gamma}_{i,\text{out}}^{\pm}} m_{\underline{\Gamma}^c})$$

the interaction energy between $\partial\underline{\Gamma}_{i,\text{out}}^{\pm}$ and $\partial\Gamma_{i,\text{in}}^{\pm}$ is not counted. ∎

Of course the configuration m^{ref} does not contain the contour Γ. It remains to find a lower bound on the energy of any configuration m that does contain a contour with given $\underline{\tilde{\Gamma}}$.

To do this, we use the following observation.

Lemma 3.8. Let $U, V, W \subset \mathbb{Z}^d$ be any three disjoint sets such that for all $y \in U \cup W$, $\sum_{x \in U \cup W \cup V} J_{\gamma l}(x-y) = 1$, and for any $y \in U$, $\sum_{x \in U \cup W} J_{\gamma l}(x-y) = 1$. Then

$$\begin{aligned} & \tilde{E}_{V \cup U \cup W}(m_{V \cup U \cup W}, m_{(V \cup U \cup W)^c}) \\ & \quad \geqslant \tfrac{1}{4} \sum_{x \in U} \psi_x(m) + \tfrac{1}{2} \sum_{x \in U \cup W} [f_\beta(m_x) + f_\beta(\phi_x(m))] + \sum_{x \in V} f_\beta(m^*) \end{aligned} \tag{3.39}$$

Proof. The proof of this lemma is a simple, but, mainly because of boundary effects, somewhat lengthy computation that we do not wish to reproduce here. To get the idea, note that in finite volume we have (formally)

$$
\begin{aligned}
-\frac{1}{2}&\sum_{x,y} m_x m_y J_{\gamma l}(x-y)+\beta^{-1}\sum_x I(m_x)\\
&=-\frac{1}{2}\sum_x m_x\phi_x(m)+\beta^{-1}\sum_x I(m_x)\\
&=\sum_x\left[\frac{(m_x-\phi_x(m))^2}{4}-\frac{m_x^2}{4}-\frac{(\phi_x(m))^2}{4}+\frac{1}{2}\beta^{-1}I(m_x)+\frac{1}{2}\beta^{-1}\phi_x(I(m))\right]
\end{aligned}
\tag{3.40}
$$

where we have put $\phi_x(I(m))=\sum_y J_{\gamma l}(x-y)\,I(m_y)$. The last line is obtained by inserting the identity $1=\sum_y J_{\gamma l}(x-y)$ in the $I(m)$ term and changing the order of summation in the resulting double sum. Using the same trick for the first term in the last line of (3.40), and using that, since I is a convex function, $\phi_x(I(m))\geqslant I(\phi_x(m))$, one gets that

$$
\sum_x\left[\tfrac{1}{4}\psi_x(m)+\tfrac{1}{2}f_\beta(\phi_x(m))+\tfrac{1}{2}f_m(m_x)\right] \tag{3.41}
$$

is a lower bound for (3.40). Trying to repeat this computation in finite volume and carefully dealing with the boundary terms leads to the more complicated-looking formula (3.39). ∎

The main point in the estimate (3.39) is that it allows us to bound the energy of a configuration from below in terms of $\phi_x(m)$ and $\psi_x(m)$ alone. Namely, taking for V and $U\cup W$ the layers $\mathscr{L}_i^\pm$ and the regions "within" $\mathscr{L}_i^\pm$, we see that for any configuration

$$
\begin{aligned}
\tilde{E}_{\underline{\Gamma}}(m_{\underline{\Gamma}}, m_{\underline{\Gamma}^c})\geqslant{}&\sum_{i,\pm}\tilde{E}_{\partial\underline{\Gamma}^\pm_{i,\mathrm{out}}}(m_{\partial\underline{\Gamma}^\pm_{i,\mathrm{out}}}, m_{\underline{\Gamma}^c})\\
&+\tfrac{1}{2}\sum_{x\in\underline{\Gamma}\backslash\partial\underline{\Gamma}_{\mathrm{out}}}\left[f_\beta(\phi_x(m))-f_\beta(m^*)\right]\\
&+\tfrac{1}{4}\sum_{\substack{x\in\underline{\Gamma}\backslash\partial\underline{\Gamma}_{\mathrm{out}}\\ \mathrm{dist}(x,\partial\underline{\Gamma}_{\mathrm{out}})>1/(\gamma l)}}\psi_x(m)\\
&+\sum_{x\in\underline{\Gamma}\backslash\partial\underline{\Gamma}_{\mathrm{out}}} f_\beta(m^*)
\end{aligned}
\tag{3.42}
$$

Next we show that both $\phi_x(m)$ and $\psi_x(m)$ have nice continuity properties.

Lemma 3.9. There exists a finite constant c_d depending only on the dimension d such that for any contour Γ, if $\hat{\underline{\Gamma}}$ denotes the set

$$\hat{\underline{\Gamma}} \equiv \left\{ y \,|\, \operatorname{dist}(y, \tilde{\underline{\Gamma}}) \leqslant \frac{(\zeta m^*)^2}{8c_d \gamma l} \right\} \tag{3.43}$$

then for all $y \in \hat{\underline{\Gamma}}$, $\|\phi_y(m)| - m^*| \geqslant \zeta m^*/2$, or $\psi_x(m) \geqslant (\zeta m^*)^2/2$.

Proof. Since $|m_x| \leqslant 1$, it is a simple geometric fact that

$$|\phi_x(m) - \phi_y(m)| \leqslant c_d \, |x-y| \, \gamma l \tag{3.44}$$

and

$$|\psi_x(m) - \psi_y(m)| \leqslant 4c_d \, |x-y| \, \gamma l \tag{3.45}$$

for some geometry-dependent constant c_d. Since on $\tilde{\underline{\Gamma}}$, $|\phi_x(m)| \geqslant \zeta m^*$ or $\psi_x(m) \geqslant (\zeta m^*)^2$, the assertion of the lemma follows. ∎

Remark. The estimates of Lemma 3.9 are very crude. In particular, we only use the trivial bound $|m_x| \leqslant 1$, whereas we would expect that, at least for configurations with low energy, it should be true that, say, $|m_x| \leqslant 2m^*$. This seems intuitively obvious, but we have not found a simple rigorous argument. To prove such a statement would considerably improve our bound on the critical temperature.

A further simple geometric consideration shows on the other hand that $\hat{\underline{\Gamma}}$ cannot to too small compared to $\underline{\Gamma}$, namely:

Lemma 3.10. There exists a numerical constant c'_d depending only on the dimension d such that for any contour Γ, we have that

$$|\underline{\Gamma}| \leqslant c'_d \frac{(n+1)^d}{(\zeta m^*)^{2d}} |\hat{\underline{\Gamma}}| \tag{3.46}$$

Proof. Note that $|\underline{\Gamma}|/|\hat{\underline{\Gamma}}|$ is maximal if $\tilde{\underline{\Gamma}}$ consists of a single point, in which case (3.46) is obvious. ∎

Combining the upper bound on the energy of m^{ref} from Lemma 3.7 with the lower bound (3.42) obtained from Lemma 3.8 applied for the optimal configuration, using the fact that that E and $\tilde{E}$ differ only by a constant that depends only on boundary conditions, and finally employing Lemma 3.10, we arrive at the following result.

Proposition 3.11. Let $\Gamma = (\underline{\Gamma}, m)$ be a contour with fixed $\underline{\Gamma}$. Then there exists a reference configuration m^{ref} in which Γ does not occur such that

$$E_{\underline{\Gamma}}(m_{\underline{\Gamma}}, m_{\underline{\Gamma}^c}) - E_{\underline{\Gamma}}(m_{\underline{\Gamma}}^{\mathrm{ref}}, m_{\underline{\Gamma}^c}^{\mathrm{ref}}) \geqslant \frac{1}{8} \frac{c(\beta)}{c_d} \frac{(\zeta m^*)^{2d+2}}{(n+1)^d} |\underline{\Gamma}| - \frac{1}{8} (m^*)^2 2^{-nc(\beta)} |\underline{\Gamma}| \tag{3.47}$$

where c_d is a finite, dimension-dependent constant and $c(\beta)$ is the constant from (3.24).

Proof. We bound $E_{\underline{\Gamma}}(m_{\underline{\Gamma}}, m_{\underline{\Gamma}^c})$ from below by the corresponding energy of the configuration m of lowest energy for given $\underline{\Gamma}$; on the belt of the contour this provides an optimal configuration in the sense of Definition 3.1. The same configuration is used in the construction of m^{ref}. After the obvious cancelations and using (3.46) and the fact that $c(\beta) \leqslant 1$, we get the assertion of the proposition. ∎

We must now begin to choose our parameters. We want the Peierls condition, i.e., that the coefficient of $|\underline{\Gamma}|$ in (3.47) is positive and as large as possible. The most convenient choice appears to be to choose n in such a way that

$$2^{-nc(\beta)} = \frac{1}{2} \frac{c(\beta)(\zeta m^*)^{2d}}{c_d(n+1)^d} \tag{3.48}$$

Calling the solution[7] of this equation n^*, we get the Peierls estimate

$$E_{\underline{\Gamma}}(m_{\underline{\Gamma}}, m_{\underline{\Gamma}^c}) - E_{\underline{\Gamma}}(m_{\underline{\Gamma}}^{\mathrm{ref}}, m_{\underline{\Gamma}^c}^{\mathrm{ref}}) \geqslant \frac{1}{16} \frac{c(\beta)(\zeta m^*)^{2d+2}}{c_d(n+1)^d} \tag{3.49}$$

It is not difficult to verify that

$$n^* \leqslant C \frac{1}{c(\beta)} \ln \left[\frac{c(\beta)(\zeta m^*)^{2d}}{2c_d} \right] \tag{3.50}$$

for some numerical constant C if $c(\beta)$ is sufficiently small.

[7] By this we will of course understand the smallest integer larger than or equal to the "real" solution.

This estimate on the energy difference will only be useful for us if it is large compared to the error terms arising from the block approximation. That is, we must make sure that

$$\frac{1}{16}\frac{c(\beta)(\zeta m^*)^{2d+2}}{c_d(n+1)^d} > c_d\gamma l \tag{3.51}$$

(the two c_d in this formula are *a priori* not the same quentities). This gives a relation between temperature-dependent quantities on the one hand and γl on the other. It does not impose any choice on the parameter l. This arises from the last condition, the comparison between the energy of a contour and the entropy, i.e., the number of configurations m on $\underline{\Gamma}$ and of shapes $\underline{\Gamma}$ with fixed volume $|\underline{\Gamma}|$. Even the crudest estimate shows that this number is smaller than $l^{d|\underline{\Gamma}|}\, C^{|\underline{\Gamma}|}$, so that (5.52) is complemented by the condition

$$\beta l^d\left[\frac{1}{16}\frac{c(\beta)(\zeta m^*)^{2d+2}}{c_d(n+1)^d} - c_d\gamma l\right] > d\ln l + \ln C \tag{3.52}$$

which requires l to be sufficiently large. In fact we may choose l as

$$l = \gamma^{-1}\frac{1}{c_d}\frac{1}{32}\frac{c(\beta)(\zeta m^*)^{2d+2}}{c_d(n+1)^d} \tag{3.53}$$

which, inserted into (3.52), gives the final condition of β in terms of γ. It is clear that for any $\beta > 1$, i.e., $c(\beta) > 0$ and $m^* > 0$, this condition can be verified by choosing γ sufficiently small. Thus, using Lemma 2.2, we proved the analog of the Peierl's argument here, namely that the probability of a given contour Γ is smaller than $\exp(-c\,|\underline{\Gamma}|\cdot|\ln l|)$, which in turn implies that the probability that the origin is in the interior of a contour is close to zero [in fact of the order $\exp(-c\beta n^d\,|\ln l|)$]. Moreover, by inserting the asymptotic behavior of m^* and $c(\beta)$, one verifies easily that if we put

$$\beta - 1 = \gamma^{(1-\varepsilon)/(2d+2)(1+1/d)} \tag{3.54}$$

for arbitrary $\varepsilon > 0$, then (3.52) is verified when γ is sufficiently small. This gives thus the claimed bound on the behavior of the critical temperature as γ tends to 0.

This concludes the proof of Theorem 1. ∎

Remark. Let us recall some consequences of what we have just proven: if V denotes the union of the interiors of all the contours of a given configuration, then the Gibbs probability of the event

$$\operatorname{dist}(i, V^c) \geqslant r \tag{3.55}$$

is independent of the choice of the point $i \in \mathbb{Z}^d$ and behaves like $\exp(-Cr)$, where $C = C(\beta, \gamma)$. This implies, for example, the following statement: The probability of the event that the support of all contours surrounding a given point is infinite is equal to zero. One could even refine such a statement, giving a more precise meaning to the intuitive idea that "almost all" configurations (of the mesoscopic observables m) in the translation-invariant + Gibbs ensemble have their local averages [in the sense of the variables $\phi_x(m)$] in the vicinity of m^* except for some (rare, but uniformly distributed throughout the lattice) "islands." (This is the appropriate rephrasing of the statement in Sinai's book.[14])

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